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n-Person Games with Only 1, $n-1$, and n-Person Permissible Coalitions*

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I. INTRODUCTION

The purpose of this paper is to present a complete description of various bargaining sets for n-person games, in which the set of permissible coalitions consists of the $1, n-1$, and n-person coalitions. These games can be regarded as special $(n-1)$-quota games (see definitions in Section III), and it turns out that the forms the bargaining sets take depend on whether or not the $(n-1)$-quota is nonnegative.

Theoretically, the bargaining sets can always be computed as solutions of certain systems of linear inequalities in the payoff space, connected by the words "and" and "or." (See Aumann and Maschler [1].) In fact, the computation of the bargaining set $\mathscr{M}$ in our case was carried out (in [1]) for $n = 3, 4$. In general, however, the computation is very difficult unless some shortcuts can be found. In this paper we used the method of "deleting" "weak players" from the original game, thus being able to obtain a very simple inductive method for constructing the bargaining sets.

It turns out that the various bargaining sets happen to be the same for the games treated here, except that, if the game has an empty core, outcomes of the form $(x_1, x_2, \cdots, x_n; 12 \cdots n)$ occur only in some of the bargaining sets.

For the sake of completeness we repeat the necessary definitions. These first appeared in [1].

II. PRELIMINARY NOTATIONS AND DEFINITIONS

We are concerned with an n-person cooperative game Γ, given by a set $N = \{1, 2, \cdots, n\}$ of n *players*, a set $\{B\}$ of nonempty subsets of N, called

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permissible coalitions, and a real valued *characteristic function* $v(B)$ defined for the elements B of $\{B\}$.

For the sake of simplicity and for normalization purposes, we assume that[1]

$$i \in \{B\}, \qquad v(i) = 0 \quad \text{for each} \quad i, \qquad i \in N. \tag{2.1}$$

$$v(B) \geq 0 \quad \text{for each} \quad B, \qquad B \in \{B\}. \tag{2.2}$$

An outcome of a game in which the players are partitioned into disjoint permissible coalitions $B_1, B_2, \cdots, B_m$, where each coalition shares its value among its members, will be described by a "payoff configuration" (p.c.):

$$(\mathbf{x}; \mathscr{B}) \equiv (x_1, x_2, \cdots, x_n; B_1, B_2, \cdots, B_m). \tag{2.3}$$

Here,

$$B_j \in \{B\}, \qquad B_j \cap B_k = \phi \quad \text{for} \quad j \neq k,$$

$$\bigcup_{j=1}^{m} B_j = N, \qquad j, k = 1, 2, \cdots, n, \tag{2.4}$$

and x_i is a real number which represents the payoff to player i, $i = 1, 2, \cdots, n$, and therefore

$$\sum_{i \in B_j} x_i = v(B_j). \tag{2.5}$$

$\mathscr{B} \equiv B_1, B_2, \cdots, B_m$ will be called the *coalition structure* and

$\mathbf{x} \equiv (x_1, x_2, \cdots, x_n)$ will be referred to as the *payoff*.

Let K be a nonempty set of players. If the players in K want to get their shares in $(\mathbf{x}; \mathscr{B})$, they need *only* the consent of the players in those coalitions which intersect K. The members of these coalitions will be called the *partners of K in* $(\mathbf{x}; \mathscr{B})$, and the set of the partners will be denoted by $P[K; (\mathbf{x}; \mathscr{B})]$. Thus

$$P[K; (\mathbf{x}; \mathscr{B})] \equiv \{i \mid i \in B_j, B_j \cap K \neq \phi\}. \tag{2.6}$$

Note that in this terminology, $K \subset P[K; (\mathbf{x}; \mathscr{B})]$. If S is a subset of a coalition B, we shall refer to the players in the complement of S with respect to B as the *associates* of S in B.

We shall be interested in "stable" payoff configurations, which will later be defined. One of the demands which we shall impose on them is that they

[1] In Section IV we shall encounter a case where the normalization $v(i) = 0$, $i = 1, 2, \cdots, n$, will not occur. Nothing in the definitions in this section should be changed if the game is not normalized.

should be *coalitionally rational*, in the sense that no subcoalition of an already existing coalition could *alone* make more. Accordingly, we make the following

DEFINITION 2.1. A p.c. (2.3) for a game Γ will be called *coalitionally rational* (c.r.p.c.) if

$$\sum_{i \in B} x_i \geq v(B) \quad \text{for all} \quad B, \qquad B \subset B_j, \qquad j = 1, 2, \cdots, m, \qquad B \in \{B\}. \tag{2.7}$$

A c.r.p.c. is certainly individually rational, but $\mathbf{x}$ in $(\mathbf{x}; \mathscr{B})$ does not have to belong to the *core* of the game, because (2.7) has to be satisfied only for each permissible B, $B \subset B_j$, $j = 1, 2, \cdots, m$.

DEFINITION 2.2. Let $(\mathbf{x}; \mathscr{B})$ be a c.r.p.c. (2.3), for a game Γ, and let K and L be two nonempty disjoint subsets of a coalition B_j which occurs in $(\mathbf{x}; \mathscr{B})$:

$$K, L \neq \phi, \qquad k \cap L = \phi, \qquad K, L \subset B_j, \qquad 1 \leq j \leq m. \tag{2.8}$$

An *objection* of K against L in $(\mathbf{x}; \mathscr{B})$ is a c.r.p.c.

$$(\mathbf{y}; \mathscr{C}) \equiv (y_1, y_2, \cdots, y_n; C_1, C_2, \cdots, C_p), \tag{2.9}$$

for which

$$P[K; (\mathbf{y}; \mathscr{C})] \cap L = \phi, \tag{2.10}$$

$$y_i > x_i \quad \text{for all} \quad i, \qquad i \in K, \tag{2.11}$$

$$y_i \geq x_i \quad \text{for all} \quad i, \qquad i \in P[K; (\mathbf{y}; \mathscr{C})]. \tag{2.12}$$

Thus, in an objection $(\mathbf{y}; \mathscr{C})$, the players in K claim that they can get more ((2.11)) without the aid of the players L ((2.10)), and the new situation is reasonable, because $(\mathbf{y}; \mathscr{C})$ is a c.r.p.c. in which the partners of K get not less than they got in $(\mathbf{x}; \mathscr{B})$ ((2.12)).

DEFINITION 2.3. Let $(\mathbf{x}; \mathscr{B})$ be a c.r.p.c. (2.3) in a game Γ, and let $(\mathbf{y}; \mathscr{C})$ be an objection of a set K against a set L in $(\mathbf{x}; \mathscr{B})$. K, L satisfy (2.8). A *counter objection* of L against K is a c.r.p.c.

$$(\mathbf{z}; \mathscr{D}) \equiv (z_1, z_2, \cdots, z_n; D_1, D_2, \cdots, D_q), \tag{2.13}$$

for which

$$P[L; (\mathbf{z}; \mathscr{D})] \not\supset K, \tag{2.14}$$

$$z_i \geq x_i \quad \text{for all} \quad i, \qquad i \in P[L; (\mathbf{z}; \mathscr{D})], \tag{2.15}$$

$$z_j \geq y_j \quad \text{for all} \quad i, \qquad i \in P[L; (\mathbf{z}; \mathscr{D})] \cap P[K; (\mathbf{y}; \mathscr{C})]. \tag{2.16}$$

In their counter objection $(\mathbf{z}; \mathscr{D})$, the players in L claim that they can protect their share ((2.15)) by giving their partners at least what they had before ((2.15)), and if they need the consent of some of the partners of K who were included in the objection, they can offer them not less than what they were offered in the objection ((2.16)). The members of L are allowed to use the tactics of "divide and rule," by taking some members of K as partners, but they may not take all the members of K as partners ((2.14)).

DEFINITION 2.4. A c.r.p.c. $(\mathbf{x}; \mathscr{B})$ in a game Γ is called *stable*, if for each objection of a set K against a set L in $(\mathbf{x}; \mathscr{B})$ there is a counter objection of L against K.

The set $\mathscr{M}$ of all the stable p.c.'s in a game Γ will be called the *bargaining set* of Γ. This bargaining set was defined and some of its properties were explored in [1].

III. EFFECTIVE COALITIONS

DEFINITION 3.1. Let B^* be a permissible coalition in a game Γ. B^* will be called an *effective coalition* if there exists a payoff $\{x_i\}$, $i \in B^*$, such that

$$\sum_{i \in B^*} x_i = v(B^*), \tag{3.1}$$

$$\sum_{i \in B} x_i \geq v(B) \quad \text{for all} \quad B, \qquad B \subset B^*, \qquad B \in \{B\}. \tag{3.2}$$

DISCUSSION. A coalition is effective if and only if its value can be shared among its members in such a way that no permissible subcoalition is able *alone* to make more. Such a share will be called an *effective share*. It follows from Definition 2.1 that only effective coalitions appear in a c.r.p.c., and the payoff of a c.r.p.c. induces an effective share in each one of its coalitions. Therefore, a noneffective coalition can never enter a stable p.c., nor can it be used for objections or counter objections. Accordingly, the bargaining set of a game Γ will not change if one declares the noneffective coalitions of Γ as not permissible.

THEOREM 3.1. *Let B^* be a permissible k-person coalition, $k \geq 2$, in a game Γ. Suppose also that any permissible subcoalition of B^* which contains at most $k - 2$ players has a zero value. Let $B^{(\nu)}$ denote the coalition $B^* - \{\nu\}$, $\{\nu\} \in B^*$, and let $v^{(\nu)} = v(B^{(\nu)})$ if $B^{(\nu)}$ is a permissible coalition, and $v^{(\nu)} = 0$ if $B^{(\nu)}$ is not permissible.*[2] *A necessary and sufficient condition for the coalition B^* to be effective is:*

$$v(B^*) \geq v^{(\nu)} \quad \text{for all} \quad \nu, \qquad \nu \in B^*, \tag{3.3}$$

$$(k - 1)\, v(B^*) \geq \sum_{i \in B^*} v^{(i)}. \tag{3.4}$$

[2] The bargaining set is not changed essentially if we replace the nonpermissible coalitions in a game by permissible coalitions having a zero value.

PROOF. If B^* is effective, let $\{x_i\}$, $i \in B^*$, be an effective share of its value among its members. Certainly $x_i \geq 0$ for all i, $i \in B^*$. Applying (3.2) to the coalitions $B^{(\nu)}$, $\nu \in B^*$, we obtain, in view of (3.1), the condition (3.3). Summing up the same inequalities (3.2) we obtain (3.4). The conditions (3.3) and (3.4) are therefore necessary.

If (3.3) and (3.4) hold, it is possible to choose numbers $w^{(\nu)}$, $\nu \in B^*$, which satisfy

$$(k-1)\, v(B^*) = \sum_{i \in B^*} w^{(i)}, \tag{3.5}$$

$$v(B^*) \geq w^{(\nu)} \quad \text{for all} \quad \nu, \qquad \nu \in B^*, \tag{3.6}$$

$$w^{(\nu)} \geq v^{(\nu)} \quad \text{for all} \quad \nu, \qquad \nu \in B^*. \tag{3.7}$$

Indeed, we increase continuously the $v^{(\nu)}$'s one by one so as not to exceed $v(B^*)$, until the sum of the new $v^{(\nu)}$'s becomes equal to $(k-1)\, v(B^*)$. We then denote these new $v^{(\nu)}$'s by $w^{(\nu)}$'s. We shall certainly reach the required sums because of (3.4) and because $kv(B^*) \geq (k-1)\, v(B^*)$. We now choose the payoffs $\{x_\nu\}$ to be:

$$x_\nu = v(B^*) - w^{(\nu)}, \qquad \nu \in B^*. \tag{3.8}$$

By (3.6), $x_\nu \geq 0$ for all ν, $\nu \in B^*$. Also, by (3.8), (3.5), and (3.6), we get for each ν, $\nu \in B^*$,

$$\sum_{i \in B^{(\nu)}} x_i = (k-1)\, v(B^*) - \sum_{i \in B^{(\nu)}} w^{(i)} = w^{(\nu)} \geq v^{(\nu)}. \tag{3.9}$$

In addition, by (3.8) and (3.5),

$$\sum_{i \in B^*} x_i = kv(B^*) - \sum_{i \in B^*} w^{(i)} = v(B^*). \tag{3.10}$$

This shows that $\{x_\nu\}$, $\nu \in B^*$, is an effective share of $v(B^*)$ among the members of B^*, hence B^* is effective. The conditions (3.3) and (3.4) are therefore sufficient.

COROLLARY 3.1. *It is easy to check that if*

$$(k-1)\, v^{(\nu)} \leq \sum_{i \in B^*} v^{(i)} \quad \text{for all} \quad \nu, \qquad \nu \in B^*, \tag{3.11}$$

then condition (3.4) implies condition (3.3). If, however, condition (3.11) is not *satisfied, then condition (3.3) implies condition (3.4).*

COROLLARY 3.2. *We shall call the conditions (3.11) generalized triangle inequalities, because of the following interesting properties:*

First, they reduce to the triangle inequalities if $k = 3$.

Secondly, if these conditions are satisfied, and B^{**} is any subset of B^*, containing r members, then the numbers $v^{(i)}$, $i \in B^{**}$, also satisfy the generalized triangle inequalities. (In particular, any three distinct numbers among the $v^{(\nu)}$'s satisfy the triangle inequalities.)

PROOF. By finite induction, it is sufficient to prove that any $k-1$ distinct numbers among the $v^{(\nu)}$'s satisfy the generalized triangle inequalities. Let B^{**} be a subset of B^* containing $k-1$ elements, and let ν_0 belong to $B^* - B^{**}$; i.e., $B^{**} = B^{(\nu_0)}$. We may write the inequality (3.11), for $\nu = \nu_0$, in the form

$$(k-2)\, v^{(\nu_0)} \leq \sum_{i \in B^{(\nu_0)}} v^{(i)}. \tag{3.12}$$

Multiplying the remaining inequalities of (3.11) by $(k-2)$ and substituting (3.12) for $(k-2)\, v^{(\nu_0)}$, we obtain

$$(k-1)(k-2)\, v^{(\nu)} \leq (k-2) \sum_{i \in B^{(\nu_0)}} v^{(i)} + \sum_{i \in B^{(\nu_0)}} v^{(i)} \tag{3.13}$$

for all ν, $\nu \neq \nu_0$, $\nu \in B^*$. Therefore,

$$(k-2)\, v^{(\nu)} \leq \sum_{i \in B^{**}} v^{(i)} \quad \text{for all} \quad \nu, \qquad \nu \in B^{**}. \tag{3.14}$$

COROLLARY 3.3. *If the generalized triangle inequalities (3.11) do not all hold, and if*

$$v^{(\nu_0)} = \operatorname*{Max}_{\nu \in B^*} v^{(\nu)}, \tag{3.15}$$

then

$$(k-1)\, v^{(\nu_0)} > \sum_{i \in B^*} v^{(i)}. \tag{3.16}$$

PROOF. If

$$(k-1)\, v^{(\nu_1)} > \sum_{i \in B^*} v^{(i)}$$

and

$$(k-1)\, v^{(\nu_2)} \leq \sum_{i \in B^*} v^{(i)},$$

then certainly

$$v^{(\nu_1)} > v^{(\nu_2)}.$$

IV. n-Person, $(n-1)$-Quota Games

Following Shapley [2] and Kalisch [3], we introduce the following:

Definition 4.1. An n-person game Γ, $n \geq 3$, is an *m-quota game*, if $2 \leq m < n$, if all the m-person coalitions are permissible and if there exist real numbers $\omega_1, \omega_2, \cdots, \omega_n$ such that[3,4]

$$v(B) = \sum_{i \in B} \omega_i \quad \text{for all} \quad B, \qquad |B| = m. \tag{4.1}$$

The vector $(\omega_1, \omega_2, \cdots, \omega_n)$ will be called the *m-quota* (or simply the quota) *of the game* Γ, and the number ω_i will be referred to as the *quota of player i*, $i = 1, 2, \cdots, n$. A player whose quota is negative (where the game is normalized by (2.1)) will be called a *weak player*. If the order of the quota is unclear from the contents, we shall say that the m-quota "contains" a weak player, or that a certain player is weak in the game with respect to the m-quota.

Lemma 4.1. *There are at most $m - 1$ weak players in an m-quota game.*

Proof. This follows immediately from (2.2) and (4.1).

Lemma 4.2. *The m-quota of a game, if such exists, is unique.*

Proof. Choose an arbitrary $(m+1)$-person subset E of N. (E exists because $m < n$). Consider the $m+1$ m-person subsets of E and apply to them the equations (4.1). The resulting linear equations in the ω_i's, $i \in E$, will have a unique solution. Thus, any $m+1$ ω_i's are uniquely determined, hence the quota is unique.

Lemma 4.3. *An n-person game, $n \geq 3$, in which the $(n-1)$-person coalitions are permissible, is an $(n-1)$-quota game. The quota is*

$$\omega_\nu = \frac{v^{(1)} + v^{(2)} + \cdots + v^{(n)} - (n-1)\, v^{(\nu)}}{n-1}, \qquad \nu = 1, 2, \cdots, n, \tag{4.2}$$

where

$$v^{(i)} \equiv v(N^{(i)}), \qquad N^{(i)} \equiv N - \{i\}, \qquad i = 1, 2, \cdots, n. \tag{4.3}$$

Proof. Equations (4.1) are satisfied.

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[3] $|B|$ denotes the number of players in B.

[4] Our definition is somewhat different than the definitions given in [2] and [3].

LEMMA 4.4. *An n-person game, regarded as an $(n-1)$-quota game, has no weak player, if and only if the values $v^{(i)}$, $i = 1, 2, \cdots, n$, defined by (4.3), satisfy the generalized triangle inequalities:*

$$(n-1)\, v^{(\nu)} \leq v^{(1)} + v^{(2)} + \cdots + v^{(n)}, \qquad \nu = 1, 2, \cdots, n. \tag{4.4}$$

(See Corollary 3.2.) The proof follows immediately from (4.2).

In this section we shall analyze n-person games in which the sets of the permissible coalitions consist of the 1, $n-1$, and n-person coalitions. We shall characterize completely the bargaining sets for these games.

THEOREM 4.1. *Let Γ be an n-person game in which the set of the permissible coalitions consists of the 1, $n-1$, and n-person coalitions. A p.c. of the form*[5]

$$(\mathbf{x}; N) \equiv (x_1, x_2, \cdots, x_n; N) \tag{4.5}$$

is stable if and only if N is an effective coalition and $\mathbf{x}$ is an effective share of $v(N)$ among the players.

PROOF. A necessary and sufficient condition that $(\mathbf{x}; N)$ is a c.r.p.c. is that N is effective and $\mathbf{x}$ is an effective share of $v(N)$ among the players. A necessary condition that $(\mathbf{x}; N)$ is stable is that it is a c.r.p.c. This condition is also sufficient, because if $(\mathbf{x}; N)$ is a c.r.p.c., no objections are possible.

COROLLARY 4.1. *If $n \geq 2$, then Theorem 3.1 gives a necessary and sufficient condition for N to be effective, namely:*

$$(n-1)\, v(N) \geq \sum_{i=1}^{n} v^{(i)}, \qquad v(N) \geq v^{(i)}, \qquad i = 1, 2, \cdots, n, \tag{4.6}$$

where $v^{(i)}$ is defined by (4.3).

The value of $v(N)$ and the fact that it is permissible or not have no effect on the stability or instability of p.c.'s of the form $(\mathbf{x}; \mathscr{B})$, where $\mathscr{B} \neq N$, because by (2.10) and (2.14), the coalition N cannot be formed in any objection or any counter objection.

Obviously, the p.c. $(0, 0, \cdots, 0;\ 1, 2, \cdots, n)$ is always stable. We have yet to find the stable p.c.'s of the form $(\mathbf{x}; N - \{j\}, j)$. It will turn out that the occurrence or nonoccurrence of a weak player in the $(n-1)$-quota plays an important role in the result.

THEOREM 4.2. *Let Γ be an n-person game, $n \geq 3$, in which the set of the permissible coalitions consists of the 1, $n-1$, and n-person coalitions. Let*

[5] "of the form" will mean in this paper "having the chosen coalition structure."

$\omega_1, \omega_2, \cdots, \omega_n$ be the $(n-1)$-quota of Γ. If no player is weak, then a p.c. of the form $\mathbf{x}; j, N^{(j)})$, where $N^{(j)} \equiv N - \{j\}$, is stable if and only if

$$(\mathbf{x}; j, N^{(j)}) = (\omega_1, \omega_2, \cdots, \omega_{j-1}, 0, \omega_{j+1}, \cdots, \omega_n; j, N^{(j)}). \tag{4.7}$$

REMARK. The quota is given by (4.2). Necessary and sufficient conditions that no player is weak are given in Lemma 4.4. This theorem asserts that if the numbers $v(N^{(j)}), j = 1, 2, \cdots, n$, satisfy the generalized triangle inequalities, then the stable p.c.'s are such that each player gets his quota if he is a member of an $(n-1)$-person coalition.

PROOF. Being an $(n-1)$-quota, the ω_i's satisfy

$$\omega_1 + \omega_2 + \cdots + \omega_{j-1} + \omega_{j+1} + \cdots + \omega_n = v^{(j)}, \qquad j = 1, 2, \cdots, n, \tag{4.8}$$

where $v^{(j)} \equiv v(N^{(j)})$. The quota for each player is nonnegative, hence (4.7) is a c.r.p.c. To show that it is stable, we observe that in any possible objection in (4.7), if any[6], a coalition $N^{(l)}$ is formed, $l \neq j$. Such an objection is an objection of a set K against player l, where K is a nonempty set of some players, each different from player j, who receive in the objection more than their quotas. The objection must therefore be of the form:

$$\begin{aligned}&(\mathbf{y}; l, N^{(l)})\\ &= (\omega_1 + \alpha_1, \omega_2 + \alpha_2, \cdots, \omega_{l-1} + \alpha_{l-1}, 0, \omega_{l+1} + \alpha_{l+1}, \cdots, \omega_n + \alpha_n; l, N^{(l)}),\end{aligned} \tag{4.9}$$

where $\alpha_i \geq 0$ for $i \neq l, j$, $\alpha_i > 0$ for $i \in K$, $l, j \notin K$, $\omega_j + \alpha_j \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_{l-1} + \alpha_{l+1} + \cdots + \alpha_n = 0$. (The last inequality follows from (4.8), where j is replaced by l.) Thus $\alpha_j < 0$.

Let μ be a player in the objecting K; then $\alpha_\mu > 0$. A counter objection can now be formed as follows[7]:

$$\begin{aligned}(\mathbf{z}; \mu, N^{(\mu)}) = (&\omega_1 + \alpha_1, \omega_2 + \alpha_2, \cdots, \omega_{l-1} + \alpha_{l-1}, \omega_l + \alpha_\mu, \omega_{l+1} + \alpha_{l+1},\\ &\cdots, \omega_{\mu-1} + \alpha_{\mu-1}, 0, \omega_{\mu+1} + \alpha_{\mu+1}, \cdots, \omega_n + \alpha_n; \mu, N^{(\mu)}).\end{aligned} \tag{4.10}$$

Evidently, $\mathbf{z} \geq 0$ coordinatewise. Also, the sum of its coordinates is $v^{(\mu)}$ (see (4.8)). Hence (4.10) is a c.r.p.c. It is a counter objection because $\alpha_\mu > 0$. We have proved that (4.7) is stable.

Suppose now that

$$(\mathbf{x}; \mathscr{B}) \equiv (x_1, x_2, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n; j, N^{(j)}) \in \mathscr{M}, \tag{4.11}$$

and suppose that $x_\mu < \omega_\mu$, $\mu \neq j$. Then, there exists at least one player ν,

[6] No objection is possible if $\omega_j = 0$.

[7] Despite the notation in (4.10), we do not assume $\mu > l$. However, $\mu \neq l$.

such that $x_\nu > \omega_\nu$. Let ϵ be a positive number, smaller than $x_\nu - \omega_\nu$, then player μ can object against player ν by[8]

$$(x_1, x_2, \cdots, x_{\mu-1}, x_\mu + \epsilon, x_{\mu+1}, \cdots, x_{j-1}, y_j, x_{j+1}, \cdots, x_{\nu-1}, 0, x_{\nu+1}, \cdots, x_n; \nu, N^{(\nu)}). \tag{4.12}$$

Here, by (4.8),

$$\begin{aligned} y_j &= v^{(\nu)} - \sum_{i \neq j, \nu} x_i - \epsilon \\ &= (\omega_1 + \omega_2 + \cdots + \omega_n - \omega_\nu) - (\omega_1 + \omega_2 + \cdots + \omega_n - \omega_j - x_\nu) - \epsilon \\ &= \omega_j + x_\nu - \omega_\nu - \epsilon > \omega_j \geq 0. \end{aligned} \tag{4.13}$$

Therefore, (4.13) is an objection.

Player ν has no counter objection. Indeed, he cannot counter object by playing as a 1-person coalition, because $x_\nu > \omega_\nu \geq 0$. He cannot appeal to the coalition N because of the restriction (2.14). By (2.14) he can only try to object by participating in the coalition $N^{(\mu)}$; but, summing up the minimum amounts that the members of this coalition must be paid in a counter objection, we obtain by (4.8), (4.11), and (4.13):

$$\begin{aligned} & x_1 + x_2 + \cdots + x_{\mu-1} + x_{\mu+1} + \cdots + x_{j-1} + y_j \\ & \quad + x_{j+1} + \cdots + x_{\nu-1} + x_\nu + x_{\nu+1} + \cdots + x_n \\ &= (\omega_1 + \omega_2 + \cdots + \omega_n - \omega_j - x_\mu) + (\omega_j + x_\nu - \omega_\nu - \epsilon) \\ &> (\omega_1 + \omega_2 + \cdots + \omega_n - \omega_\mu) + (x_\nu - \omega_\nu - \epsilon) \\ &= v^{(\mu)} + (x_\nu - \omega_\nu - \epsilon) > v^{(\mu)}. \end{aligned}$$

This sum is greater than the amount the coalition $N^{(\mu)}$ can make, hence a counter objection is not possible.

We have proved that $(\mathbf{x}; \mathscr{B})$ is not stable, contrary to the assumption (4.11), hence each player in $(\mathbf{x}; \mathscr{B})$ must receive at least, and therefore exactly, his quota. This completes the proof of the theorem.

If a weak player is present in the $(n - 1)$-quota, we shall see that the game is essentially reduced to a similar $(n - 1)$-person game. To this effect, we shall now explain how to "delete" players from our games.

DEFINITION 4.2. Let Γ be an n-person game, $n \geq 3$, in which the set of the permissible coalitions consists of the 1, $n - 1$, and n-person coalitions. A game Γ_ν is said to result from game Γ by *deleting player* ν, if:

(i) The players in Γ_ν form the set $N - \{\nu\}$.

[8] Despite the notation in (4.12), we assume no order relation among the numbers μ, ν, j, except that they are distinct.

(ii) The set of permissible coalitions in Γ_ν consists of the 1, $n-2$, and $(n-1)$-person coalitions.

(iii) The characteristic function $v_\nu(B)$ of Γ_ν satisfies

$$v_\nu(B^{(\nu)}) = v(B), \tag{4.14}$$

whenever B is an $(n-1)$-person coalition in Γ and $B^{(\nu)} \equiv B - \{\nu\}$ and, if $n > 3$,

$$v_\nu(i) = 0 \quad \text{for all} \quad i, \qquad i \in N - \{\nu\}. \tag{4.15}$$

DISCUSSION. Note that $B^{(\nu)} = B$ if $\nu \notin B$. In deleting player ν from the game Γ, we ignored the n-person coalition, and from each $(n-1)$-person coalition which contained player ν we made an (n − 2)-person coalition by removing this player from the coalition without altering the value of the coalition. The $(n-1)$-person coalition which did not include player ν remained unchanged. If $n = 3$, it may well happen that the characteristic function does not satisfy the normalization (2.1).

LEMMA 4.5. *If* $\{\omega_i\}$, $i = 1, 2, \cdots, n$ *is the* $(n-1)$-*quota of* Γ, *then* $\{\omega_i + \omega_\nu/(n-2)\}$, $i = 1, 2, \cdots, \nu-1, \nu+1, \cdots, n$ *is the* $(n-2)$-*quota of* Γ_ν.

The proof is immediate.

COROLLARY 4.2. *If we order the players by their quotas, then deleting a player from the game will not change the order relation for the rest of the players. In particular, if we delete a weak player, the remaining weak players (if such exist) will remain weak, some previously nonweak players may now become weak, but their new quotas will be greater than the new quotas of the previously weak players.*

LEMMA 4.6. *Let* Γ *be an n-person game, in which the set of the permissible coalitions consists of the* 1, $n-1$, *and n-person coalitions. Let*

$$(\mathbf{x}; j, N^{(j)}) \equiv (x_1, x_2, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n; j, N - \{j\}) \tag{4.16}$$

be a c.r.p.c. in Γ. *Then:*

(i) *An objection of the form*

$$(\mathbf{y}; l, N^{(l)}) \equiv (y_1, y_2, \cdots, y_{l-1}, 0, y_{l+1}, \cdots, y_n; l, N - \{l\}) \tag{4.17}$$

exists[9] *if and only if* $l \neq j$ *and*

$$\sum_{i \neq j, l} x_i < v^{(l)}. \tag{4.18}$$

[9] By (2.10) and (2.11), any possible objection must be of this form.

(ii) *If an objection of the form (4.17) exists, for a fixed l, $l \neq j$, then each objection of this form can be countered, if and only if*

$$x_l = 0 \quad \text{or} \quad v^{(k)} + x_k \geq v^{(l)} + x_l \quad \text{for all} \quad k, \quad k \neq j. \tag{4.19}$$

Here

$$v^{(\nu)} \equiv v(N^{(\nu)}) \equiv v(N - \{\nu\}), \qquad \nu = 1, 2, \cdots, n.$$

PROOF. The necessity and sufficiency of condition (4.18) follows immediately from Definition 2.2. Suppose that (4.18) is satisfied, and that any objection of the form (4.17) can be countered, for a fixed l. Each objection of this form must be regarded as an objection of a set K against player l, where K is a nonempty subset of the set consisting of those players, different from player j, for which $y_i > x_i$. In particular, for each fixed player k, $k \in N - \{l\} - \{j\}$, we can distribute the amount $v^{(l)}$ in such a way that $y_k = x_k + \epsilon$, where ϵ is an arbitrarily chosen small enough positive number, and make this distribution an objection of player k against player l. Player l can counter object in at most two ways: either by playing as a 1-person coalition, which is possible if and only if $x_l = 0$; or by joining the coalition $N^{(k)}$, which is possible if and only if

$$v^{(k)} - x_l \geq v^{(l)} - x_k - \epsilon. \tag{4.20}$$

Thus, either $x_l = 0$ must hold or (4.20), for each small enough positive ϵ. We have therefore proved that (4.19) is a necessary condition.

Conversely, if $x_l = 0$, player l can always counter object by playing as a 1-person coalition. If (4.19) is satisfied, but $x_l > 0$, let $(\mathbf{y}; l, N^{(l)})$ be any objection of K against l. Certainly, K is a non-empty subset of $N - \{l\} - \{j\}$. Let k be a player in K, then $y_k > x_k$. We claim that player l has a counter objection against K, of the form[10]

$$(\mathbf{z}; k, N^{(k)}) \equiv (y_1, y_2, \cdots, y_{k-1}, 0, y_{k+1}, \cdots, y_{l-1}, z_l, y_{l+1}, \cdots, y_n; k, N^{(k)}), \tag{4.21}$$

where

$$z_l = v^{(k)} - \sum_{i \neq k, l} y_i. \tag{4.22}$$

Indeed, because of (4.19),

$$z_l = v^{(k)} - v^{(l)} + y_k > v^{(k)} - v^{(l)} + x_k \geq x_l. \tag{4.23}$$

This completes the proof.

[10] Despite the notation in (4.21), l is not necessarily greater than k.

THEOREM 4.3. *Let Γ be an n-person game, $n \geq 3$, in which the set of the permissible coalitions consists of the* 1, $n-1$, *and n-person coalitions. Suppose that the $(n-1)$-quota of the game contains a weak player, and let ν be the player having the smallest quota. Let Γ_ν be the $(n-1)$-person game which results by deleting player ν from Γ. Under these conditions:*

(i) *A p.c. of the form*

$$(\mathbf{x};\, \nu,\, N^{(\nu)}) \equiv (x_1, x_2, \cdots, x_{\nu-1}, 0, x_{\nu+1}, \cdots, x_n;\, \nu,\, N - \{\nu\}) \tag{4.24}$$

is stable in Γ, if and only if

$$(\mathbf{x}^*;\, N^{(\nu)}) \equiv (x_1, x_2, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_n;\, N - \{\nu\}) \tag{4.25}$$

is stable in Γ_ν. Such stable p.c.'s always exist.

(ii) *A p.c. of the form*

$$(\mathbf{x};\, j,\, N^{(j)}) \equiv (x_1, x_2, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n;\, j,\, N - \{j\}), \qquad j \neq \nu, \tag{4.26}$$

is stable in Γ if and only if[11] $x_\nu = 0$ *and*

$$(\mathbf{x}^*;\, j,\, N^{(\nu,j)}) \equiv (x_1, x_2, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n;\, j,\, N - \{\nu\} - \{j\} \tag{4.27}$$

is stable in Γ_ν.

PROOF. (i) Player ν has the smallest quota; hence, by (4.1) and (4.14),

$$v_\nu \equiv v^{(\nu)} = \operatorname*{Max}_{i \in N} v^{(i)} \geq \operatorname*{Max}_{i \in N - \{\nu\}} v^{(i)} = \operatorname*{Max}_{i \in N - \{\nu\}} v_\nu{}^{(i)}, \tag{4.28}$$

where the following notation is used for the characteristic function $v_\nu(B)$ of Γ_ν:

$$v_\nu \equiv v_\nu(N - \{\nu\}), \qquad v_\nu{}^{(i)} \equiv v_\nu(N - \{\nu\} - \{i\}) \quad \text{for} \quad i \neq \nu. \tag{4.29}$$

By Corollary 3.3,

$$(n-1)\, v_\nu = (n-1)\, v^{(\nu)} > \sum_{i \in N} v^{(i)} = v_\nu + \sum_{i \in N - \{\nu\}} v_\nu{}^{(i)}. \tag{4.30}$$

Therefore,

$$(n-2)\, v_\nu > \sum_{i \in N - \{\nu\}} v_\nu{}^{(i)}. \tag{4.31}$$

Corollary 4.1, applied to the game Γ_ν, (4.28) and (4.31) insure the effectiveness[12] of the coalition $N - \{\nu\}$ in Γ_ν. Thus, by Theorem 4.1, stable p.c.'s for the game Γ_ν, of the form (4.25) always exist.

[11] Despite the notation in (4.27), ν is not necessarily smaller than j.

[12] Even if $n = 3$, where (2.1) does not necessarily hold for Γ_ν.

If (4.25) is stable in the game Γ_ν, then certainly (4.24) is a c.r.p.c., because $v_\nu = v^{(\nu)}$, and because $\mathbf{x}^* \geq 0$ coordinatewise. We know already that 1 and n-person coalitions cannot be used in any objection to (4.24). The p.c. (4.25), being stable in Γ_ν, induces by Theorem 4.1 an effective share of v_ν among the players of $N - \{\nu\}$. This means that

$$\sum_{i \neq \nu, l} x_i \geq v_\nu^{(l)} \equiv v^{(l)} \quad \text{for each} \quad l, \qquad l \neq \nu. \tag{4.32}$$

Lemma 4.6, part (i), now implies that no objections are possible in (4.24), with respect to the game Γ. Therefore (4.24) is stable in Γ.

Conversely, if (4.25) is not stable in Γ_ν, but (4.24) is stable in Γ, then, by Theorem 4.1, $\mathbf{x}^*$ is not an effective share[13] of v_ν among the players in Γ_ν. Therefore, there exists a coalition $N^{(\nu,l)}$, $\nu \neq l$, for which

$$\sum_{i \neq \nu, l} x_i < v_\nu{}^{(l)} = v^{(l)}. \tag{4.33}$$

Lemma 4.6, part (i), now asserts that there exists an objection of the form (4.17) in (4.24), with respect to the game Γ. If we show that (4.19) is not satisfied, then, by Lemma 4.6, part (ii), (4.25) would not be stable in Γ, whence a contradiction. This is indeed the case, because if $x_l = 0$, then (4.33) would imply $v_\nu < v^{(l)}$, contradicting (4.28). If for each k, $k \neq \nu$,

$$v^{(k)} + x_k \geq v^{(l)} + x_l,$$

then (4.33) would imply

$$v^{(k)} + x_k > \sum_{i \neq \nu, l} x_i + x_l = \sum_{i \neq \nu} x_i = v^{(\nu)}. \tag{4.34}$$

Summing up these inequalities for $k = 1, 2, \cdots, \nu - 1, \nu + 1, \cdots, n$, we obtain

$$\sum_{i=1}^{n} v^{(i)} > (n-1)\, v^{(\nu)}. \tag{4.35}$$

This contradicts (4.30).

(ii) Consider the expressions (4.26), with $x_\nu = 0$, and (4.27), with respect to the games Γ and Γ_ν, respectively. Certainly, if one of them is a p.c. in its game, so is the other. Also, they are simultaneously coalitionally or non-coalitionally rational in their respective games.[14] We can therefore assume that (4.26) and (4.27) are both c.r.p.c. in their respective games. Applying

[13] Even in the case $n = 3$, where the assumption (2.1) does not necessarily hold for Γ_ν.

[14] One has to check especially the case $n = 3$, when the assumption (2.1) does not necessarily hold for Γ_ν.

Lemma 4.6 to the game Γ, we find that the p.c. (4.26) with $x_\nu = 0$ is stable in Γ if and only if for each l, $l \neq j$,

$$\sum_{i \neq j, l} x_i \geq v^{(l)} \quad \text{or} \quad x_l = 0 \quad \text{or} \quad v^{(k)} + x_k \geq v^{(l)} + x_l \quad \text{for all} \quad k, \quad k \neq j. \tag{4.36}$$

Applying the same Lemma to game Γ_ν, and taking (4.14) into account, we find that the p.c. (4.27) is stable in Γ_ν if and only if for each l, $l \neq j$, ν,

$$\sum_{i \neq j, l, \nu} x_i \geq v^{(l)} \quad \text{or} \quad x_l = 0 \quad \text{or} \quad v^{(k)} + x_k \geq v^{(l)} + x_l \text{ for all } k, \quad k \neq j, \nu. \tag{4.37}$$

Certainly, (4.36) implies (4.37), because $x_\nu = 0$. We shall show that the converse is also true! Indeed, (4.36) is always satisfied for $l = \nu$. Thus, if (4.37) is satisfied but (4.36) does not hold, then a player l, $l \neq j$, ν exists, for which,

$$\sum_{i \neq j, l} x_i < v^{(l)}, \qquad x_l > 0, \qquad \text{and} \qquad v^{(\nu)} + x_\nu < v^{(l)} + x_l. \tag{4.38}$$

Using again the fact that $x_\nu = 0$, we find that

$$v^{(\nu)} < v^{(l)} + x_l \leq v^{(k)} + x_k \quad \text{for all} \quad k, \qquad k \neq j, \nu. \tag{4.39}$$

Summing up these inequalities for $k \neq j$, ν, we obtain

$$(n-2)\, v^{(\nu)} < \sum_{i \neq j, \nu} v^{(i)} + \sum_{i \neq j} x_i = \sum_{i \neq \nu} v^{(i)}, \tag{4.40}$$

or

$$(n-1)\, v^{(\nu)} < \sum_{i=1}^{n} v^{(i)}. \tag{4.41}$$

This contradicts (4.30).

We have proved that the p.c.'s (4.26) with $x_\nu = 0$, and (4.27) are simultaneously stable or unstable in their respective games. It remains to be shown that the p.c. (4.26) is never stable if[15] $x_\nu > 0$. We shall show that then (4.36) is not satisfied for $l = \nu$.

It follows from (4.28) that

$$\sum_{i \neq j, \nu} x_i = v^{(j)} - x_\nu < v^{(j)} \leq v^{(\nu)}. \tag{4.42}$$

If

$$v^{(k)} + x_k \geq v^{(\nu)} + x_\nu \quad \text{for all} \quad k, \qquad k \neq j, \tag{4.43}$$

[15] It is certainly unstable if $x_\nu < 0$, because then (4.26) is not c.r.

then, replacing x_ν in (4.43) by 0 and summing these inequalities, we obtain

$$\sum_{i \neq j} v^{(i)} + v^{(j)} = \sum_{i=1}^{n} v^{(i)} > (n-1)\, v^{(\nu)}. \tag{4.44}$$

This contradicts (4.30). Thus, (4.26) cannot be stable if $x_\nu > 0$, which brings the proof of the theorem to its end.

CONCLUSION. The procedure for obtaining the bargaining set for the games described in this section is set by Theorems 4.1, 4.2, and 4.3, as follows:

The p.c. $(0, 0, \cdots, 0; 1, 2, \cdots, n)$ is always stable. Theorem 4.1 and Corollary 4.1 provide the stable p.c.'s of the form $(\mathbf{x}; N)$. These exist if and only if the game has a nonempty core, and the payoffs in the stable p.c.'s form exactly the core of the game.

If the generalized triangle inequalities (4.4) hold for the values of the $(n-1)$-person coalition, i.e., if no weak player exists in the $(n-1)$-quota, then Theorem 4.2 provides us with all the stable p.c.'s of the form $(\mathbf{x}; j, N - \{j\})$.

If there is a weak player in the game, we delete the weakest player ν (i.e., the player with the smallest quota) and reduce the game to an $(n-1)$-person game with the 1, $n-2$, and $(n-1)$-person coalitions as permissible. The new game has a nonempty core. The stable p.c. of the form $(\mathbf{x}^*; j, N - \{j\} - \{\nu\})$ in the new game are in $1-1$ correspondence to those stable in the original game, of the form $(\mathbf{x}; j, N^{(j)})$. To obtain a stable p.c. in the original game, we pick a stable p.c. in the reduced game, we add the weak player with a 0 payoff, and change the coalition structure in an obvious way.

If, by deleting the weakest players, one at a time, we finally arrive at an m-person game, $m \geq 3$, with no weak player in the $(m-1)$-quota, we calculate by Theorems 4.1 and 4.2 its bargaining set, from which we can successively return to the original game, each time by using Theorem 4.1.

If we arrive at a 3-person game, which still contains a weak player, then reducing once more we arrive at a 2-person game of the players, say, k and l, with the characteristic function

$$v(k) = a, \qquad v(l) = b, \qquad v(kl) = c, \qquad c > a + b. \tag{4.45}$$

Obviously, the bargaining set for this game consists of the p.c.'s:

$$(a, b; k, l), \quad (a + \alpha, b + \beta; kl), \quad \alpha, \beta \geq 0, \quad \alpha + \beta = c - a - b. \tag{4.46}$$

From this bargaining set we go back, as before, to the stable p.c.'s of the form $(\mathbf{x}; j, N - \{j\})$ in the original game.

Two features should be stressed:

1. For any coalition-structure of the form j, $N - \{j\}$, there always corresponds a payoff $\mathbf{x}$ such that $(\mathbf{x}; j, N - \{j\})$ is stable. The same holds true for the coalition-structure N, if and only if N is an effective coalition.

2. The game has a discrete bargaining set if and only if its $(n - 1)$-quota contains no weak player and the n-person coalition is either ineffective or its value is exactly equal to the sum of the quotas of the players.

V. Other Kinds of Bargaining Sets

The bargaining set $\mathscr{M}$ represents possible outcomes of a game, assuming that the players wish to end up with that kind of stability which is implied by the definition of $\mathscr{M}$. Such a wish may sometimes cause inconvenience to the players: It may happen, for example, that the n-person coalition is not effective, yet its value is greater than the value of any other coalition.[16] One may argue that the players might be willing to relax their stability requirements somewhat, in order to take advantage of forming the n-person coalition.

Taking another point of view, one may argue that an outcome in $\mathscr{M}$ is only protected against threats *within* an existing coalition, but not against objections raised by members in *several* existing coalitions against other players in these same coalitions.

Aumann and Maschler [1] suggested several possible modifications of the theory for coping with such situations. In this section, we shall examine how some of these modifications affect the bargaining set for the games which were treated in the previous section.

Definition 5.1. A c.r.p.c. $(\mathbf{x}; \mathscr{B})$ will be called $\mathscr{M}_0$*-stable* if it satisfies Definition 2.4, where objections and counter objections are given by Definitions 2.2 and 2.3, except that (2.8) is now replaced by

$$K, L \neq \phi, \quad K \cap L = \phi, \quad K \cap B_j \neq \phi \Leftrightarrow L \cap B_j = \phi, \quad \text{for} \quad B_j \in \mathscr{B}. \tag{5.1}$$

Verbally, we are now dropping the restriction that K and L belong to one coalition B_j, requiring instead that K and L intersect the *same* coalitions in $\mathscr{B}$.

The set of all the $\mathscr{M}_0$-stable p.c.'s will be called the *bargaining set* $\mathscr{M}_0$.

Certainly, $\mathscr{M}_0 \subset \mathscr{M}$, and this inclusion may indeed be strict (see [1, 4]).

In this section, in order to be specific, we shall refer to p.c.'s in $\mathscr{M}$ as $\mathscr{M}$*-stable* p.c.'s.

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[16] This happens, e.g., if $v(N)$ in essential constant-sum games (with a superadditive characteristic function) is slightly increased.

THEOREM 5.1. *Let Γ be an n-person game, in which the set of the permissible coalitions consists of the* $1, n-1$, *and n-person coalitions. The bargaining sets* $\mathscr{M}$ *and* $\mathscr{M}_0$ *for* Γ *are equal.*

PROOF. In any possible coalition-structure $\mathscr{B}$, there occurs at most one coalition which contains more than one player. By (5.1), neither K nor L may contain a 1-person coalition of $\mathscr{B}$, whence, in our games, condition (5.1) is equivalent to condition (2.8).

DEFINITION 5.2. A c.r.p.c. $(\mathbf{x}; \mathscr{B})$ will be called $\mathscr{M}_1$-*stable* if for each objection of a set K against a set L, there is at least one player in L who can counter-object. (Objections and counter objections are being taken as in[17] Definitions 2.2 and 2.3.)

The set of all the $\mathscr{M}_1$-stable p.c.'s will be called the *bargaining set* $\mathscr{M}_1$.

Since an objection of a set K against a set L may be regarded also as an objection of K against any particular member of L, the condition for $\mathscr{M}_1$-stability is equivalent to requiring that *each* player in L is able himself to counter-object (although the players in L may find it impossible to counter-object collectively). Thus, $\mathscr{M} \subset \mathscr{M}_1$, and this inclusion may indeed be strict (see [1]). We further note that any objection of a set of players K against a single player l may be regarded as an objection of a single player k in K, against player l. In view of (2.14), one observes that Definition 5.2 is equivalent to Definition 2.4, when restricting K and L, in the latter case, to containing only *one* player.

THEOREM 5.2. *Let Γ be an n-person game, in which the set of the permissible coalitions consists of the* $1, n-1$, *and n-person coalitions. The bargaining sets* $\mathscr{M}$ *and* $\mathscr{M}_1$ *are the same for* Γ.

PROOF. It is sufficient to prove that $\mathscr{M}_1 \subset \mathscr{M}$ for our games.

Certainly, there is only one p.c. with the coalition structure $1, 2, \cdots, n$, and it is $\mathscr{M}_1$ as well as $\mathscr{M}$-stable.

If $(x_1, x_2, \cdots, x_n; N)$ is $\mathscr{M}_1$-stable, then it is *a fortiori* a c.r.p.c., and hence, by Theorem 4.1, it is also $\mathscr{M}$-stable.

Suppose that $(\mathbf{x}; j, N^{(j)}) \equiv (x_1, x_2, \cdots, x_n; j, N - \{j\})$ is $\mathscr{M}_1$-stable. Let $\mathbf{y}; \mathscr{C})$ be an objection of a set K against a set L, in $(\mathbf{x}; j, N^{(j)})$. Clearly, $j \notin K, L$ and therefore $\mathscr{C}$ must be of the form $l, N^{(j)}, l \neq j$. Thus L consists of *one* player, namely player l, and by $\mathscr{M}_1$-stability, the objection can indeed be countered.

DEFINITIONS 5.3, 5.4, 5.5. The *bargaining sets* $\mathscr{M}^{(i)}, \mathscr{M}_0^{(i)}$ and $\mathscr{M}_1^{(i)}$ are defined exactly as $\mathscr{M}$, $\mathscr{M}_0$, and $\mathscr{M}_1$, respectively, except that the coalitional

[17] Although Theorem 5.2 will remain valid if we replace (2.8) by (5.1).

rationality restriction imposed on the p.c.'s is replaced everywhere by the requirement that the p.c.'s are *individually rational.* I.e., we replace (2.7) by the weaker requirement[18]:

$$x_i \geq 0 \quad \text{for all} \quad i, \qquad i = 1, 2, \cdots, n. \tag{5.2}$$

Clearly, $\mathscr{M}_0^{(i)} \subset \mathscr{M}^{(i)} \subset \mathscr{M}_1^{(i)}$. An example in[19] [4] shows that the first inclusion may indeed be strict. The following example will show that also the second inclusion may be strict.

EXAMPLE 5.1.[20] $n = 4$, $\{B\} = \{i, ij, 123\}$, $v(i) = 0$, $v(ij) = 10$, $v(123) = 18$ for all i, j where $i, j = 1, 2, 3, 4$ and $i \neq j$. Clearly $(6, 6, 6, 0; 123, 4)$ belongs to $\mathscr{M}_1^{(i)}$ but not to $\mathscr{M}^{(i)}$.

We shall see later[21] that each of the bargaining sets $M_0^{(i)}, \mathscr{M}^{(i)}$ and $\mathscr{M}_1^{(i)}$ may be different from $\mathscr{M}_0$, $\mathscr{M}$, and $\mathscr{M}_1$.

THEOREM 5.3. *Let Γ be an n-person game, in which the set of the permissible coalitions consists of the* 1, $n - 1$, *and n-person coalitions. The bargaining sets* $\mathscr{M}_0^{(i)}, \mathscr{M}^{(i)}$, *and* $\mathscr{M}_1^{(i)}$ *are the same for Γ. Moreover, the p.c.'s in these bargaining sets which are not of the form* $(\mathbf{x}; N)$ *coincide with those in* $\mathscr{M}$.

PROOF. Any i.r.p.c. not of the form $(\mathbf{x}; N)$ must be also coalitionally rational. P.c.'s of the form $(\mathbf{x}; N)$ never enter into objections or counter objections. This, and the fact that $\mathscr{M}_0 = \mathscr{M} = \mathscr{M}_1$, assures the last assertion.

Any objection in $(\mathbf{x}; N)$, in any of the above definitions, must be of the form $(\mathbf{x}; l, N^{(l)})$, and any counter objection to such an objection, in any of the above definitions, must either satisfy $x_l = 0$ or be of the form $(\mathbf{x}; k, N^{(k)})$, where player k belongs to the objecting K. Thus, one sees easily that the conditions that $(\mathbf{x}; N)$ is $\mathscr{M}_0^{(i)}$, $\mathscr{M}^{(i)}$, $\mathscr{M}_1^{(i)}$-stable are equivalent. This completes the proof.

We realize that the only new p.c.'s which might appear in $\mathscr{M}^{(i)}$ and not in $\mathscr{M}$ are p.c.'s of the form $(\mathbf{x}; N)$. We shall see that this will actually happen. It is remarkable, however, that $\mathscr{M}^{(i)}$-stable p.c.'s of the form $(\mathbf{x}; N)$, which are not $\mathscr{M}$-stable, occur if and only if N is not an effective coalition. In other words, as long as $\mathscr{M}$-stable p.c.'s of the form $(\mathbf{x}; N)$ exist, all the bargaining sets considered here are identical for our games. If no $\mathscr{M}$-stable p.c. of this form exists, the players may relax their stability requirements, by replacing the coalitional rationality requirement by individual rationality restriction. Doing so, they will *always* find an $\mathscr{M}^{(i)}$-stable p.c. of the form $(\mathbf{x}; N)$.

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[18] If (2.1) is not assumed, we require $x_i \geq v(i)$ for all i.

[19] Example 3.1. In this example coalitional rationality coincides with individual rationality.

[20] A similar example was suggested to me by H. Kesten, in a different context.

[21] Two examples of this kind were indicated also in [1].

In order to prove this and analyze the new p.c.'s, we shall first establish the following:

LEMMA 5.1. *Let Γ be the game described in Theorem 5.3. A p.c. of the form* $(\mathbf{x}; N)$ *is* $\mathscr{M}^{(i)}$ *stable, if and only if for each player* l, $l = 1, 2, \cdots, n$,

$$\sum_{i \neq l} x_i \geq v^{(l)} \quad \text{or} \quad x_l = 0 \quad \text{or} \quad v^{(k)} + x_k \geq v^{(l)} + x_l \text{ for all } k. \tag{5.3}$$

Here, $v^{(\nu)} \equiv v(N^{(\nu)}) \equiv v(N - \{\nu\})$, $\nu = 1, 2, \cdots, n$.

The proof of this lemma is similar to the proof of Lemma 4.6, and will therefore be omitted.

THEOREM 5.4. *Let Γ be an n-person game, $n \geq 2$, in which the set of permissible coalitions consists of the* 1, $n - 1$, *and n-person coalitions. If N is an effective coalition, then a p.c. of the form* $(\mathbf{x}; N)$ *is* $\mathscr{M}^{(i)}$*-stable if and only if it is* $\mathscr{M}$*-stable.*

PROOF. If $(\mathbf{x}; N)$ is $\mathscr{M}$-stable, then it is a c.r.p.c., and the first inequality in (5.3) holds for each l. Hence $(\mathbf{x}; N)$ is also $\mathscr{M}$-stable.

Conversely, suppose that $(\mathbf{x}; N)$ is $\mathscr{M}^{(i)}$-stable. This means that (5.3) holds. If there exists a player μ for which

$$\sum_{i \neq \mu} x_i < v^{(\mu)} \quad \text{and} \quad x_\mu = 0, \tag{5.4}$$

then

$$v(N) = \sum_{i=1}^{n} x_i = \sum_{i \neq \mu} x_i < v^{(\mu)}. \tag{5.5}$$

By Corollary 4.1, N is not an effective coalition, contrary to our assumption. If there exists a player μ for which

$$\sum_{i \neq \mu} x_i < v^{(\mu)} \quad \text{and} \quad v^{(k)} + x_k \geq v^{(\mu)} + x_\mu \quad \text{for all} \quad k, \tag{5.6}$$

then

$$v(N) = \sum_{i=1}^{n} x_i < v^{(\mu)} + x_\mu \leq v^{(k)} + x_k \quad \text{for all} \quad k. \tag{5.7}$$

Summing up these inequalities for $k = 1, 2, \cdots, n$, we obtain:

$$(n - 1)\, v(N) < \sum_{k=1}^{n} v^{(i)}. \tag{5.8}$$

Again, by Corollary 4.1, N is not an effective coalition, contrary to our assumption. Thus, the first inequality in (5.3) must hold for each l, whence, by Theorem 4.1, $(\mathbf{x}; N)$ is $\mathscr{M}$-stable. This completes the proof.

The situation is more complicated if N is not an effective coalition. Several cases will be distinguished.

THEOREM 5.5. *Let Γ be an n-person game, $n \geq 3$, in which the set of the permissible coalitions consists of the* 1, $n-1$, *and n-person coalitions. Let N be a noneffective coalition. If the p.c.*

$$(\omega_1 - c, \omega_2 - c, \cdots, \omega_n - c; N), \tag{5.9}$$

where

$$c = \frac{\omega_1 + \omega_2 + \cdots + \omega_n - v(N)}{n}, \tag{5.10}$$

is individually rational,[22] *then this p.c. is $\mathscr{M}^{(i)}$-stable, and no other p.c. of the form $(\mathbf{x}; N)$ is $\mathscr{M}^{(i)}$-stable. In particular, (5.9) is always individually rational if the characteristic function of the game Γ is superadditive.*[23]

PROOF. *Case A.* Suppose that

$$v(N) \geq v(N - \{\nu\}) \equiv v^{(\nu)} \quad \text{for each} \quad \nu, \qquad \nu = 1, 2, \cdots, n. \tag{5.11}$$

By Corollary 4.1, since N is not an effective coalition, we obtain:

$$(n-1)\, v(N) < \sum_{i=1}^{n} v^{(i)}. \tag{5.12}$$

Therefore,

$$(n-1)\, v^{(\nu)} < \sum_{i=1}^{n} v^{(i)} \quad \text{for all} \quad \nu, \qquad \nu = 1, 2, \cdots, n, \tag{5.13}$$

which, by (4.8), implies

$$(n-1) \sum_{i \neq \nu} \omega_i < \sum_{i=1}^{n} \sum_{j \neq i} \omega_j = (n-1) \sum_{i=1}^{n} \omega_i. \tag{5.14}$$

We see that $\omega_\nu > 0$ for all ν, $\nu = 1, 2, \cdots, n$; hence,

$$\begin{aligned} \omega_\nu - c &= \frac{n\omega_\nu + v(N) - \omega_1 - \omega_2 - \cdots - \omega_n}{n} \\ &= \frac{(n-1)\, \omega_\nu + v(N) - v^{(\nu)}}{n} > 0. \end{aligned} \tag{5.15}$$

[22] I.e., if $\omega_\nu - c \geq 0$ for each ν, $\nu = 1, 2, \cdots, n$.

[23] I.e., if (5.11) holds.

This proves that (5.9) is indeed an i.r.p.c. Since (5.9) is an i.r.p.c., then, by (5.3), it is $\mathscr{M}^{(i)}$-stable, because for each k and l,

$$v^{(k)} + (\omega_k - c) = \sum_{i=1}^{n} \omega_i - c = v^{(l)} + (\omega_l - c). \tag{5.16}$$

Conversely, let $(\mathbf{x}; N)$ be an $\mathscr{M}^{(i)}$-stable p.c. for Γ, then (5.3) is satisfied. Since, by Theorem 4.1, the game has an empty core, there exists a player $l = \mu$, for which the first inequality of (5.3) does not hold. We know that (5.4) implies (5.5), which is contrary to (5.11), hence we must conclude that $x_\mu > 0$, and that

$$v^{(k)} + x_k \geq v^{(\mu)} + x_\mu \quad \text{for all} \quad k, \qquad \sum_{i \neq \mu} x_i < v^{(\mu)}. \tag{5.17}$$

If, for some player ρ we had

$$\sum_{i \neq \rho} x_i \geq v^{(\rho)}, \tag{5.18}$$

we would obtain

$$v(N) = \sum_{i=1}^{n} x_i \geq v^{(\rho)} + x_\rho \geq v^{(\mu)} + x_\mu > \sum_{i=1}^{n} x_i = v(N), \tag{5.19}$$

which is a manifest contradiction. Thus, (5.18) never holds, $x_\nu > 0$ for each player ν; and hence, by (5.3)

$$v^{(k)} + x_k = v^{(l)} + x_l, \qquad \sum_{i=1}^{n} x_i = v(N) \tag{5.20}$$

holds for each k and l. Equations (5.20) have a unique solution; therefore $(\mathbf{x}; N)$ is the p.c. (5.9).

Case B. We now suppose that (5.11) does not hold. The fact that (5.9) is an i.r.p.c. is by (4.2) and (5.15) equivalent to

$$v(N) + \sum v^{(i)} \geq nv^{(\nu)} \quad \text{for all} \quad \nu, \qquad \nu = 1, 2, \cdots, n. \tag{5.21}$$

As before, (5.9) is certainly $\mathscr{M}^{(i)}$-stable because (5.16) holds.

Let $(\mathbf{x}; N)$ be an $\mathscr{M}^{(i)}$-stable p.c. for Γ, then (5.3) is satisfied. Let S be the set of indices ν, for which

$$\sum_{i \neq \nu} x_i < v^{(\nu)}, \tag{5.22}$$

and let $R = N - S$. S is not empty, because the game has an empty core.

Obviously,

$$v(N) = \sum_{i=1}^{n} x_i < v^{(\nu)} + x_\nu \quad \text{for each} \quad \nu, \qquad \nu \in S, \tag{5.23}$$

$$v(N) = \sum_{i=1}^{n} x_i \geq v^{(\mu)} + x_\mu \quad \text{for each} \quad \nu, \qquad \nu \in R. \tag{5.24}$$

I. We shall examine now the case in which $x_\nu = 0$ for each ν, $\nu \in S$. In this case $v(N) < v^{(\nu)}$ for each ν, $\nu \in S$. If[24] $|R| \neq 0$, we can sum up the inequalities (5.24), and we obtain

$$(|R| - 1)\, v(N) \geq \sum_{i \in R} v^{(i)}. \tag{5.25}$$

Let $v^{(\alpha)} = \mathrm{Max}\,(v^{(1)}, v^{(2)}, \cdots, v^{(n)})$. Certainly, $\alpha \in S$, because $v^{(\alpha)} > v(N)$. From (5.21) we now obtain

$$nv^{(\alpha)} \leq v(N) + \sum_{i \in S} v^{(i)} + (|R| - 1)\, v(N) \leq |R|\, v(N) + |S|\, v^{(\alpha)}, \tag{5.26}$$

which implies $v^{(\alpha)} \leq v(N)$. This, however, cannot happen, because we are dealing with the case in which (5.11) does not hold.

Thus, $R = \phi$, hence $v(N) = 0$. In this case, $(\mathbf{x}; N)$ must be $(0, 0, \cdots, 0; N)$. We shall show that it must be equal to the p.c. (5.9). This follows easily from (5.10) and from the fact that (5.9) is an individually rational p.c.

II. Suppose that there exists a player ρ, $\rho \in S$, for which $x_\rho \neq 0$. Certainly, $v(N) > 0$. By (5.3), then

$$v^{(k)} + x_k \geq v^{(\rho)} + x_\rho \quad \text{for each} \quad k, \qquad k = 1, 2, \cdots, n. \tag{5.27}$$

Again, $R = \phi$, since otherwise, for a player σ in R we would have obtained by (5.23), (5.24) and (5.27) the impossible inequality:

$$v(N) \geq v^{(\sigma)} + x_\sigma \geq v^{(\rho)} + x_\rho > v(N). \tag{5.28}$$

Thus, $S = N$. Let P be the set of players τ for which $x_\tau = 0$, and let $v^{(\alpha)} = \mathrm{Max}\,(v^{(1)}, v^{(2)}, \cdots, v^{(n)})$. Let Q be the set of players ρ for which $x_\rho > 0$. We know that $Q \neq \phi$ and that (5.27) holds for each ρ, $\rho \in Q$; hence

$$v^{(\tau)} \geq v^{(\rho)} + x_\rho \quad \text{whenever} \quad \tau \in P, \qquad \rho \in Q. \tag{5.29}$$

Summing up these relations for all the ρ's in Q, we obtain

$$|Q|\, v^{(\tau)} \geq \sum_{i \in Q} v^{(i)} + v(N) \quad \text{whenever} \quad \tau \in P, \tag{5.30}$$

[24] $|P|$ denotes the number of elements in a set P.

which implies

$$v(N) + \sum_{i \in Q} v^{(i)} + \sum_{i \in P} v^{(i)} \leq |Q| v^{(\tau)} + \sum_{i \in P} v^{(i)} \leq nv^{(\alpha)}. \quad (5.31)$$

By (5.21) we realize that this situation can happen only if $v^{(\tau)} = v^{(\alpha)}$ for each τ, $\tau \in P$, and if

$$v(N) + \sum v^{(i)} = nv^{(\tau)} \quad \text{whenever} \quad x_\tau = 0. \quad (5.32)$$

It follows that $\omega_\tau - c = 0$ whenever $x_\tau = 0$. The inequalities (5.20), restricted to all k, l, where $k, l \in Q$, determine uniquely x_ρ for each ρ, $\rho \in Q$, hence $(\mathbf{x}; N)$ is the p.c. (5.9). This completes the proof of the theorem.

If (5.9) is not an i.r.p.c., it cannot be $\mathcal{M}^{(i)}$-stable. In order to get the stable p.c.'s in this case, one has to refer to some sort of "deleting" of players from the game. This will be established as follows:

LEMMA 5.2. *Let Γ be a game as described in Theorem 5.5, except that (5.9) be no longer individually rational. Let α be any one of the players having the smallest quota, and let $(\mathbf{x}; N)$ be an $\mathcal{M}^{(i)}$-stable p.c. Under these conditions, $x_\alpha = 0$.*

PROOF. We know that in this case the characteristic function of the game is not superadditive (see end of Theorem 5.5). Certainly $\omega_\alpha - c < 0$. If $x_\alpha > 0$, then, by (5.3),

$$\text{either} \quad \sum_{i \neq \alpha} x_i \geq v^{(\alpha)} \qquad \text{or} \qquad v^{(k)} + x_k \geq v^{(\alpha)} + x_\alpha \quad \text{for each} \quad k. \quad (5.33)$$

By (4.2), $v^{(\alpha)} = \text{Max}\,(v^{(1)}, v^{(2)}, \cdots, v^{(n)})$; hence the first inequality in (5.33) implies $v(N) \geq v^{(\alpha)} + x_\alpha > v^{(\alpha)}$, which is impossible because (5.11) does not hold, and the second inequality in (5.33) implies

$$\sum_{i=1}^{n} v^{(i)} + v(N) \geq nv^{(\alpha)} + nx_\alpha > nv^{(\alpha)}, \quad (5.34)$$

which is again impossible because it implies (see (5.21)) $\omega_\alpha - c > 0$. Therefore $x_\alpha = 0$.

THEOREM 5.6. *Let Γ be an n-person game, $n > 3$, in which the set of permissible coalitions consists of the 1, $n - 1$, and n-person coalitions. Let α be a player having the smallest quota, and let $\omega_\alpha - c < 0$, where c is defined by (5.10). Let Γ^* be an $(n - 1)$-person game which results from Γ by deleting* *the player* α *and assigning the remaining players the following characteristic function:*

$$v^*(B) = \begin{cases} v(B \cup \{\alpha\}) \text{ whenever } B = N - \{\alpha\} - \{\nu\}, \ \nu = 1, 2, \cdots, n, \ \nu \neq \alpha, \\ v(N) \quad \text{for} \quad B = N - \{\alpha\}, \\ 0 \quad \text{for} \quad B = 1, 2, \cdots, \alpha - 1, \alpha + 1, \cdots, n. \end{cases} \tag{5.35}$$

Under these conditions,

$$(\mathbf{x}; N) \equiv (x_1, x_2, \cdots, x_n; N) \tag{5.36}$$

is $\mathscr{M}^{(i)}$*-stable with respect to* Γ, *if and only if* $x_\alpha = 0$ *and*

$$(\mathbf{x}^*; N^*) \equiv (x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots, x_n; N - \{\alpha\}) \tag{5.37}$$

is $\mathscr{M}^{(i)}$*-stable with respect to* Γ^*.

PROOF. We know already that $x_\alpha = 0$ is a necessary condition for $(\mathbf{x}; N)$ to be $\mathscr{M}^{(i)}$-stable with repect to Γ, hence (5.3) reduces to

$$\sum_{i \neq \nu, \alpha} x_i \geq v^{(\nu)} \quad \text{or} \quad x_\nu = 0 \quad \text{or} \quad v^{(k)} + x_k \geq v^{(\nu)} + x_\nu \quad \text{for each} \quad k, \tag{5.38}$$

which are to be satisfied for each ν, $\nu \neq \alpha$, $\nu = 1, 2, \cdots, n$. The corresponding conditions for $(\mathbf{x}^*; N^*)$ to be $\mathscr{M}^{(i)}$-stable with respect to Γ^* are *the same inequalities* (5.38), except that k is not required to cover player α. The theorem will be proved if we can show that no player φ exists, for which

$$v^{(k)} + x_k \geq v^{(\varphi)} + x_\varphi \quad \text{for each} \quad k, \quad k \neq \alpha, \qquad v^{(\alpha)} + x_\alpha < v^{(\varphi)} + x_\varphi, \tag{5.39}$$

f $(\mathbf{x}; N)$ is an i.r.p.c. with respect to Γ. Indeed, otherwise, we obtain

$$v^{(k)} + x_k > v^{(\alpha)} \quad \text{for each} \quad k, \quad k \neq \alpha, \tag{5.40}$$

and hence

$$\sum_{i=1}^{n} v^{(i)} + v(N) > nv^{(\alpha)}. \tag{5.41}$$

This contradicts the fact that $\omega_\alpha - c < 0$ (see (5.21)).

DISCUSSION. The $\mathscr{M}^{(i)}$-stable p.c.'s of the form $(\mathbf{x}; N)$ are the core of the game, provided that the core is not empty. If the core is empty but (5.9) is an i.r.p.c., it is also the unique $\mathscr{M}^{(i)}$-stable p.c. of this form. If this is not the case, in order to obtain the $\mathscr{M}^{(i)}$-stable p.c. of the form $(\mathbf{x}; N)$, one "deletes" the "weakest" players one at a time, as explained in Theorem 5.6, until one arrives at a game for which the $\mathscr{M}^{(i)}$-stable p.c.'s are known. The players which are deleted would get a zero payoff anyway, and the payoff for the others is determined to be their payoffs in the $\mathscr{M}^{(i)}$-stable p.c.'s in the reduced game. Our analysis will be completed if we list the $\mathscr{M}^{(i)}$-stable p.c.'s for the 3-person game with an empty core, for which (5.9) is not individually rational:

THEOREM 5.7. *Let Γ be a 3-person game, the characteristic function of which satisfies*

$$v^{(1)} \geq v^{(2)} \geq v^{(3)}, \qquad v(N) < v^{(1)}, \qquad v(N) + v^{(2)} + v^{(3)} < 2v^{(1)}, \quad (5.42)$$

where $v^{(i)} = v(N - \{i\})$, $i = 1, 2, 3$. The following are the necessary and sufficient conditions for a p.c. $(\mathbf{x}; N) \equiv (x_1, x_2, x_2; 123)$ to be $\mathscr{M}^{(i)}$-stable with respect to Γ:

(i) *If* $v(N) \geq v^{(2)} + v^{(3)}$ *then* $x_1 = 0$, $x_2 \geq v^{(3)}$, $x_3 \geq v^{(2)}$ *and* $x_2 + x_3 = v(N)$.

(ii) *If* $v(N) < v^{(2)} + v^{(3)}$ *but* $v(N) + v^{(3)} \geq v^{(2)}$ *and* $v(N) + v^{(2)} \geq v^{(3)}$, *then* $x_1 = 0$, $x_2 = \frac{1}{2}(v(N) + v^{(3)} - v^{(2)})$, $x_3 = \frac{1}{2}(v(N) + v^{(2)} - v^{(3)})$.

(iii) *If* $v(N) + v^{(3)} - v^{(2)} < 0$, *then* $x_1 = x_2 = 0$, $x_3 = v(N)$.

The proof follows immediately from analyzing the inequalities (5.3).

COROLLARY 5.1. For our games, there always exists an $\mathscr{M}^{(i)}$-stable p.c. of the form $(\mathbf{x}; N)$.

COROLLARY 5.2. The procedure described in Section IV, of deleting a weak player from the game in order to find the $\mathscr{M}$-stable p.c.'s of the form $(\mathbf{x}; j, N^{(j)})$, is the same procedure for finding the $\mathscr{M}^{(i)}$-stable p.c.'s of the same form. In other words, when we treat the reduced game, we may look for the $\mathscr{M}^{(i)}$-stable p.c.'s for this game, and forget about the $\mathscr{M}$-stability theory. This is a consequence of the fact that the reduced game always has a nonempty core (see end of case (i) in Theorem 4.3 and the conclusion at the end of Section IV). For games with a nonempty core, $\mathscr{M} = \mathscr{M}^{(i)}$.

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